Getting Your Hands Dirty in Integral Calculus

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Abstract

The landscapes of many elementary, middle, and high school math classrooms have undergone major transformations over the last half-century, moving from drill-and-skill work to more conceptual reasoning and hands-on manipulative work. However, if you look at a college level calculus class you are likely to find the main difference is the professor now has a whiteboard marker in hand rather than a piece of chalk. It is possible that some student work may be done on the computer, but much of it contains the same type of repetitive skill building problems. This should seem strange given the advancements in technology that allow more freedom than ever to build connections between different representations of a concept.

Several class activities have been developed using a combination of approaches, depending on the topic. All activities use student note outlines that are either done in a whole group interactive-lecture approach, or in a group work discovery-based approach. Some of the activities use interactive graphs designed on desmos.com and others use physical models that have been designed in OpenSCAD and 3D-printed for students to use in class. Tactile objects were developed because they should provide an advantage to students by enabling them to physically interact with the concepts being taught, deepening their involvement with the material, and providing more stimuli for the brain to encode the learning experience. Web-based activities were developed because the topics involved needed substantial changes in graphical representations (i.e. limits with Riemann Sums).

Topics covered in the activities include Riemann Sums, Accumulation, Center of Mass, Volumes of Revolution (Discs, Washers, and Shells), and Volumes of Similar Cross-section. Assessment techniques for these topics include online homework, exams, and online concept questions with an explanation response area. These concept questions are intended to measure students' ability to use multiple representations in order to answer the question, and are not generally computational in nature. Students are also given surveys to rate the overall activities as well as finer grained survey questions to try and elicit student thoughts on certain aspects of the models, websites, and activity sheets. We will report on student responses to the activity surveys, looking for common themes in students' thoughts toward specific attributes of the activities. We will also compare relevant exam question responses and online concept question results, including common themes present or absent in student reasoning.

Introduction

We don't pay attention to boring things. Stimulate more of the senses. We are powerful and natural explorers. These are just a few of the dozen statements describing how our brains work in John Medina's book "Brain Rules" [1]. These also provide insight into the fundamental reasons for our Integral Calculus reforms developed over the last year.

We don't pay attention to boring things. Math courses are notorious for being boring, lecture-heavy periods of time that are unavoidable since they are necessary for earning a degree. Over the years, several interventions have described ways to transform math classrooms into a place where students are actively engaged and not drooling on their desks or intently gazing at the back of their eyelids. Flipped Classrooms [2], Inquiry Based Learning [3], Interactive Lectures with Clicker Questions or visualization software [4] [5], and Active Learning [6] [7] are just some of the more recent interventions a select number of math faculty are trying. Watson, et al., provide an excellent overview of calculus course interventions that have been reported through ASEE from 2005 to 2018 [8]. Most of these interventions are designed to help improve student success or retention in future engineering courses. Some of these interventions include direct changes within calculus courses, while others add extra experiences outside of the usual classroom period.

A key feature of our intervention in Calculus II is the flexibility of implementation. Changes to courses were two-fold. The first is physical models or websites that enhance student interaction with the concepts being explored. The second is student note/activity sheets that can be used in a variety of ways, depending on the topic and how the instructor wants to structure the course. These can be used as guides for students to fill out independently in groups, or as guided lecture notes to be completed as a class. These will be described in more detail in the next section, with samples provided and a link to the rest of the guided notes. We will also explore student reactions to the interactive websites, models, and activity sheets.

Stimulate more of the senses. There is a reason that so many elementary school classrooms have physical manipulatives to help teach math. Not only do they counteract boredom, but they also give a concrete representation of a more abstract concept. The student that uses these tactile tools is in a better position to encode the learning experience in a deeper way. Yet we rarely find tactile manipulatives in Calculus courses. Some of this may be due to a lack of availability of models at this level, and some may be due to a lack of interest from faculty.

We are powerful and natural explorers. How our brains learn and encode things into memory is complex, and often different in various individuals [1]. To complicate matters further, the way we are presented information can also be a help or a hindrance. Graphs, pictures, diagrams, equations, and language are all important aspects of mathematical and general STEM ideas. Our ability to "see through" the surface features of a representation to the underlying concept has been referred to as representational competence (RC) in chemistry education literature [9]. The construct of RC has also been applied in education research in other science fields [10] [11], engineering [12] [13], and mathematics [14]. Experts demonstrate the ability to freely move between different representations of an idea, knowing which ones are most useful, and which do not provide relevant insights. Novices can sometimes translate from one representation to

another, but they typically have trouble integrating the information from these various sources into a larger cohesive framework. Studies in chemistry education demonstrate the promise of tactile models and computer simulations to help scaffold students' development of RC [15], [16].

We have leveraged the relatively recent boom in 3D printing technology to create physical manipulatives to use in the integral calculus classroom. These tactile tools are designed to be used in conjunction with activity sheets, and help provide a concrete focal point for students to "get their hands dirty" while investigating the concepts being learned. Our intervention aims to help students become more aware of the different representations in ideas through the use of the activity sheets, technology, physical models, and different types of assessment questions. Several "concept check" questions were given to students as a means of testing their conceptual understanding and representational competence, rather than just rote procedural calculations. Some of these questions were given online as a pre-test "readiness check", and others were given on the exams. Questions typically consisted of a multiple choice portion, followed by an explanation portion where students justified their responses.

Topics covered in the activities include Riemann Sums, Accumulation, Center of Mass, Volumes of Revolution (Discs, Washers, and Shells), and Volumes of Similar Cross-section. Assessment techniques for these topics include online homework, exams, and online concept questions with an explanation response area. In this paper, we provide examples of activities and assessment data primarily from the Volumes of Revolution topic that is in integral calculus, mainly to keep things concise and thematic. Links to activity sheets and associated .stl files for 3D printing are available here: https://graspthemath.wordpress.com/integral-calculus/ [17]. Although all of the examples in this paper are from integral calculus, a similar approach of using activity sheets, physical manipulatives, and concept check questions could be applied to other STEM courses.

Guided Notes with Electronic Aides

Some topics lend themselves to visual representations that involve real time animations or manipulations to change specific aspects of the image. For instance, Figure 1 shows a Desmos page created for the purpose of investigating Riemann Sums. When investigating Riemann sums in terms of area under a curve, we often estimate the area using rectangles with either right-handed sums, left-handed sums, or midpoint sums, and the limit of these sums as the number of rectangles increases. It is quite time consuming to do many of these by hand, and yet it is beneficial for students to physically draw a sample of these by hand to get a sense of how they work and how the graphs connect to more symbolic representations. Creating graphical representations through Desmos or other software can provide a way to quickly manipulate the different aspects of the rectangles typically used to explain Riemann sums.

Although the "area under a curve" interpretation of an integral is the most commonly used graphical representation, it gives only limited insight into the Fundamental Theorem of Calculus and the Net Change Theorem. An accumulation approach to Riemann sums is another graphical approach that more closely ties together the ideas of sums and antiderivatives, and gives a much richer insight into these theorems. Figure 2 shows an interactive Desmos webpage for an accumulation approach to Riemann sums, which is often left out of calculus textbooks.
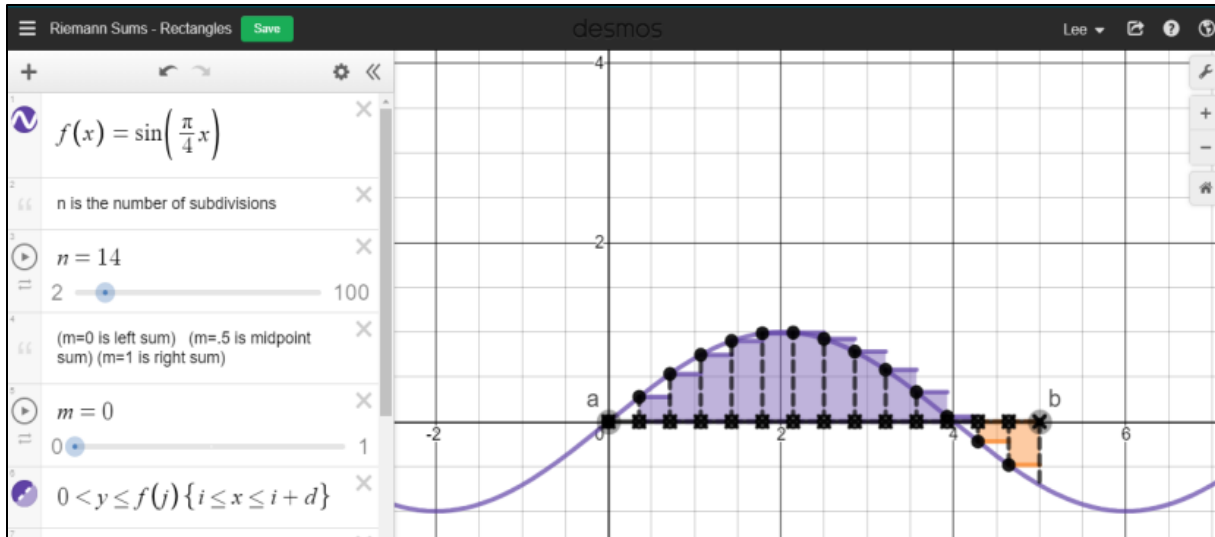
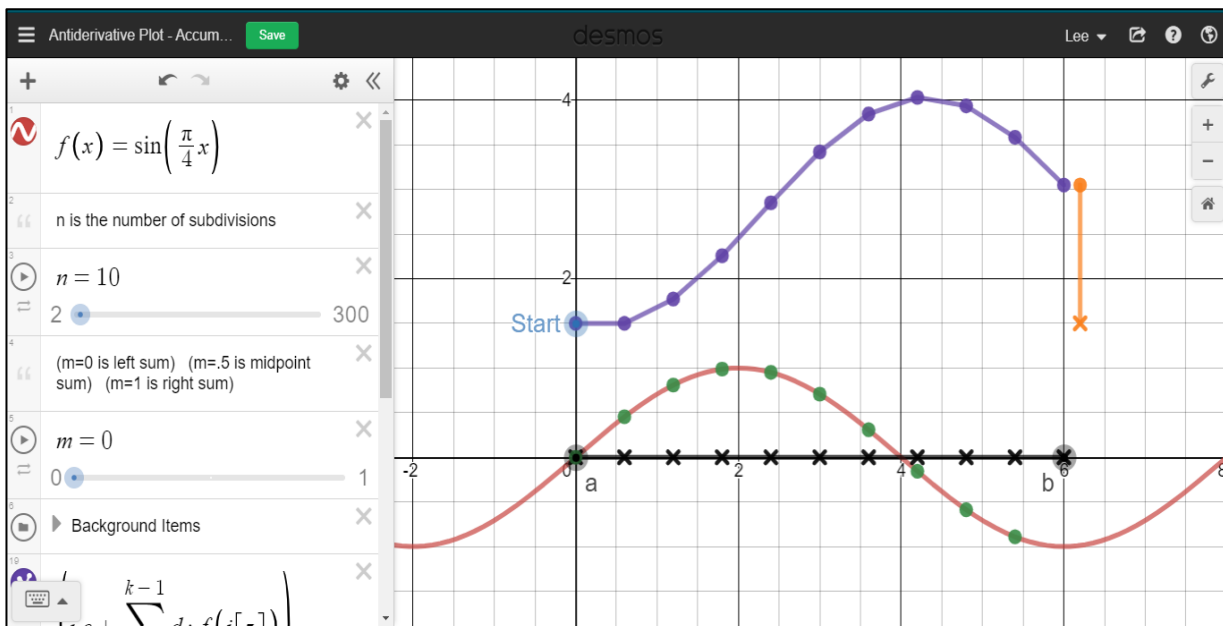*Figure 1: Desmos page using an interactive graph to illustrate rectangles for Riemann Sums. This interactive graph is available at https://www.desmos.com/calculator/s0gswsulii.*

Figure 2: Interactive Desmos site for interpreting Riemann Sums as accumulation. This interactive graph is available at https://www.desmos.com/calculator/cjedeenx5f.

These topics naturally lend themselves to using interactive online graphs, but the activities still have a physical element of sketching by hand. Blending activity sheets and interactive graphs allows students to focus on the animations in the graphs as well as scaffolding the main ideas they should include in their notes. The activity sheet prompts students to draw several graphs by hand on given Cartesian planes. They also answer conceptual questions based on changes to the interactive graphs. The full activity sheet is included in the appendix.

Activity Sheets and Physical Manipulatives

Integral Calculus is full of topics that seem to present students with significant difficulty. In particular, solids of revolution often seem particularly troublesome for students, probably due to the spatial visualization required to understand the problems [18]. By pairing activity sheets with physical models that students can grasp and explore, we hope to enhance student spatial visualization skills as well as improve their understanding of certain topics such as solids of revolution (discs, washers, shells). The volumes of revolution models, shown in Figure 3, can be taken apart so that students can manipulate the region being rotated as well as a sample washer or shell. A sample activity for volumes of revolution is located in the appendix.
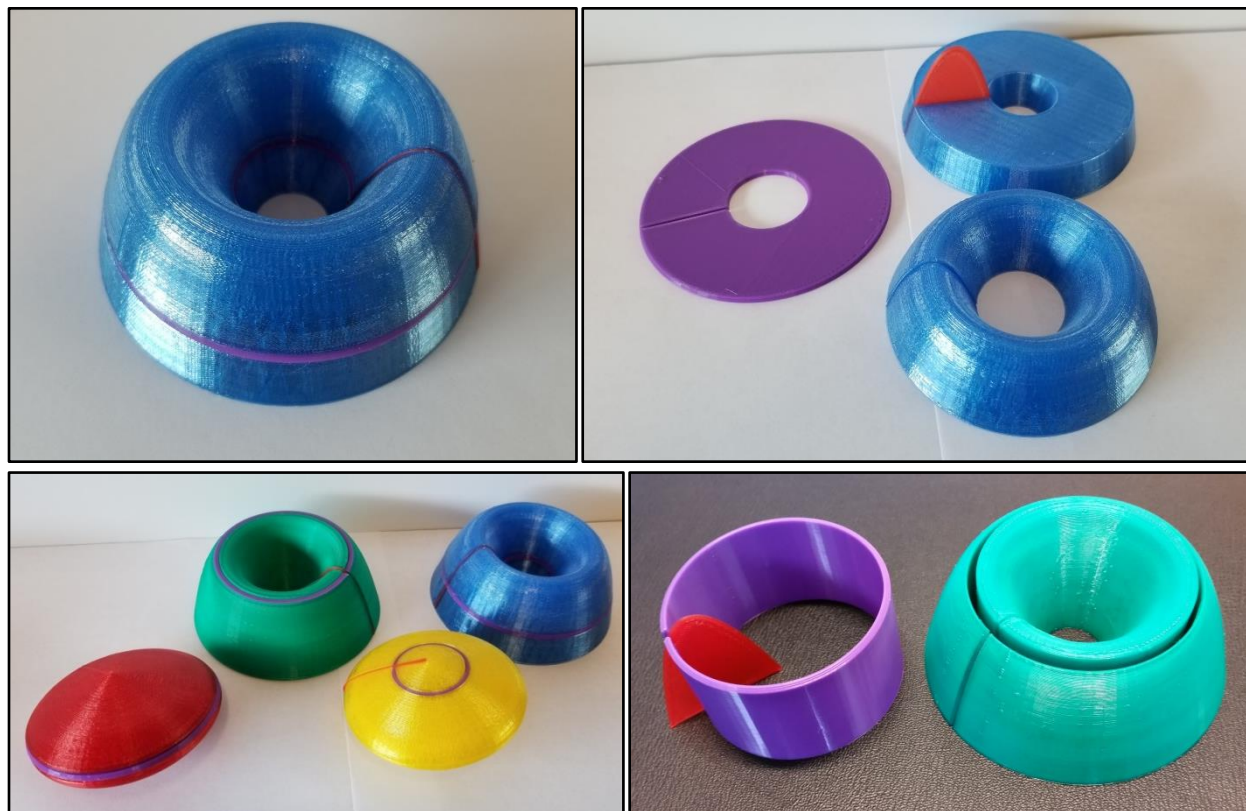

Figure 3: Top and Bottom Left – Volume of Revolution Models, used for investigating the washer method. Top and Bottom Right – Volume Model taken apart to reveal the region being rotated and a sample washer or shell.

All models were developed in OpenSCAD and 3D-printed prior to students using the models and activities in class. Several activities and models for centroids, volumes of revolution, and volumes of similar cross-section were developed, and all activities and links to the .stl files for the models can be found here: https://graspthemath.wordpress.com/integral-calculus/ [17]. Please note, some of these activities are still undergoing revisions, so updates to the website can be expected as improvements are made.

Implementation of the activities with models is typically begun with a brief instructor-led introduction to the topic for about 5-10 minutes. Then students are given time to work on the activities in groups while the instructor floats around the room answering questions and checking student work to make sure they are on the right track. The instructor will also do a "whole class check" a few times throughout the lesson, asking for answers to a set of questions that students should have completed at that point. These "checks" serve as a way for students to verify that they are on the right track in terms of accuracy and time management. Typical class sessions in this intervention are only 50 minutes, and classes are held five days per week for the quarter (11-12 weeks). Initially, the short time frame allotted each day was a challenge, with activities usually taking longer than planned. As the activities are refined, the time estimates for each activity are becoming more accurate, and tend to fit well into the 50 minute time periods. For longer class sessions, multiple activities could be used together.

Time always seems to be a big factor when deciding whether or not to try an intervention. The same amount of time was spent on each topic for classes using models vs. non-models since each course went through similar activity sheets. Active learning can take a little more time than just lectures. For example, four days were spent on volumes of revolution using activity sheets (with models or electronic visuals), versus three days typically spent using lecture only. However, most topics used a similar amount of time using activity sheets and models as opposed to lecture.

Concept Check Questions

It is our hope that students will also improve their conceptual understanding of the topics they are learning, and not just rote procedures for doing problems that they then regurgitate on exams [19]. In order to facilitate assessing students understanding of certain concepts, new multiple choice questions were developed through spring and fall quarters of 2019. These multiple choice questions were given in two avenues – concept check questions given online prior to the exam and questions given on regular in-class exams. All of these questions have a multiple choice component and an answer justification component. Figure 4 shows two sample questions that were given on the topics of Riemann sums and centroids.
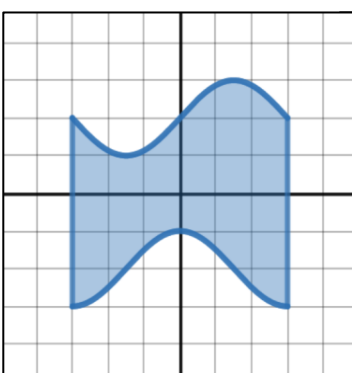

The function $f(x)$ is strictly increasing on the entire interval $[0, 5]$.

Which of the following is in order from smallest to largest?

- $L_4 < \int_0^5 f(x)dx < R_8 < R_4$
- $R_8 < R_4 < \int_0^5 f(x)dx < L_4$
- $L_4 < \int_0^5 f(x)dx < R_4 < R_8$
- $R_4 < R_8 < \int_0^5 f(x)dx < L_4$

Please explain your answer in the box below.

A region with uniform density is shown in the figure

Where would you find the centroid of the region?

- Origin
- Positive x-axis
- Negative x-axis
- Positive y-axis
- Negative y-axis
- Not on any axes

Please explain your answer in the box below.

Figure 4: Two "Concept Check" questions that are used to gain insight into student thinking and their line of reasoning while problem-solving.

The online questions given as a "concept check" prior to exams were awarded full credit for completion, and not for accuracy. These questions were talked about in class a day or two before the exam to allow students to see where they may have applied incorrect reasoning and other avenues they might have taken to answer the question. The exam questions were given on exams with one point counting toward the multiple choice portion of the answer and five points toward the justification portion of the answer. Points for the justification portion were awarded as seen in Table 1. Table 2 shows sample student responses for the question in Figure 5. These are representative responses for each score from the rubric.

Points	Justification
5	Complete answer with enough reasoning to justify the correct response.
4	Complete answer with a slight error in reasoning or calculation. Most of the response still contains appropriate elements.
3	Complete answer with a major error in reasoning or calculation. Part of the response would contain appropriate elements.
2	Incomplete answer with partially correct reasoning, or significant errors in reasoning with a small portion that contains correct elements.
0	Missing justification, or a completely incorrect response.

Table 1: Sample rubric for assigning points to "concept check" questions on exams. One point is awarded for a correct multiple choice answer, and up to 5 points for the justification.

Points	Justification
5	The more of the region R that is further from the point of rotation, the larger the solid it will make.
4	Because a majority of B's region has a longer distance around the axis, the volume is larger and A is smaller. Knowing this, C has to be the biggest because of how far the region is from its axis of rotation.
3	With C it is rotating around y=4, which will create the most overlap, which will make it smaller. A is the next smallest, rotating around x=2. That will make a cup shape. The largest is B, rotating around y=-5. This will create the biggest range and will cover the most of the graph.
2	Since A's shape involves a larger area when turned around x = 2, it gives the largest volume. Since C has a larger radius than B, it has the larger volume making B the smallest volume.
0	The size of the hole in the washer determines the area since the large area is the same no matter what because the area of the shape doesn't change.

Table 2: Sample rubric for assigning points to "concept check" questions on exams. One point is awarded for a correct multiple choice answer, and up to 5 points for the reasoning.

Unlike standard multiple choice questions, these do a tremendous job at highlighting where a student is correct or incorrect in their conceptual understanding. This method of assessment can really highlight where a student is making errors in their thinking, and where they may be on track with the relevant concepts. A comparison of student responses will be conducted in the future on control and intervention sections, to see where student thinking may or may not differ based on whether or not they used the physical manipulatives.

An analysis of six concept check questions from the volumes unit was done to see how students perform on these types of questions. Two were given prior to the third exam (CC1 and CC3), and four were given on the third exam (T3-1, 2, 3, 4). The first pre-exam question (CC1) is shown in Figure 5. The second pre-exam question and the four test questions are shown below in Figure 6.
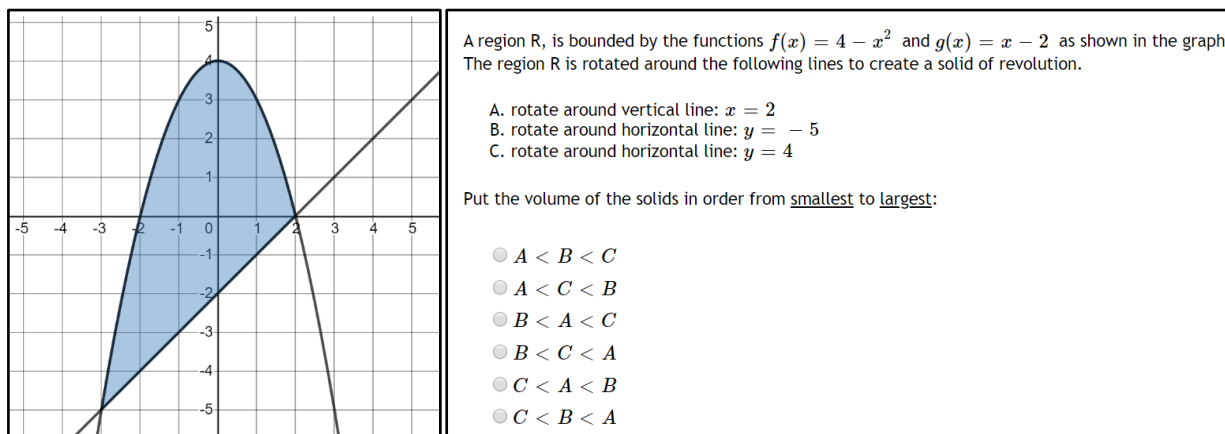
A region R, is bounded by the functions $f(x) = 4 - x^2$ and $g(x) = x - 2$ as shown in the graph The region R is rotated around the following lines to create a solid of revolution.

A. rotate around vertical line: $x = 2$
B. rotate around horizontal line: $y = -5$
C. rotate around horizontal line: $y = 4$

Put the volume of the solids in order from smallest to largest:

○ $A < B < C$
○ $A < C < B$
○ $B < A < C$
○ $B < C < A$
○ $C < A < B$
○ $C < B < A$

Figure 5: "Concept Check" online question 1 for the volumes unit (CC1).

The region R shown in the graph above is rotated around the y-axis to generate a solid of revolution. A student decides to start finding the volume by dividing the region using vertical rectangles. Which method is being used to find the volume of the solid?

○ Disc Method, integrating with respect to x
○ Disc Method, integrating with respect to y
○ Shell Method, integrating with respect to x
○ Shell Method, integrating with respect to y

1. (1 pt) The region, R, shown is rotated around the line $y = 7$. Which integral would represent the volume of the resulting solid?

a) $\pi \int_1^5 (f(x))^2 - (g(x))^2 \, dx$

b) $\pi \int_1^5 (7 - g(x))^2 - (7 - f(x))^2 \, dx$

c) $2\pi \int_1^5 x (f(x) - g(x)) \, dx$

d) $2\pi \int_1^5 (7 - x)(f(x) - g(x)) \, dx$

(5 pts) Explain Your Reasoning:

2. (1 pt) A region, R, lies in Quadrant I with a single point touching the y-axis. R is then rotated around the y-axis to generate a solid. If a student uses horizontal strips, which of the following are they using to find the volume of the solid?

a) Discs, integrating with respect to x
b) Discs, integrating with respect to y
c) Washers, integrating with respect to x
d) Washers, integrating with respect to y
e) Cylindrical Shells, integrating with respect to x
f) Cylindrical Shells, integrating with respect to y

(5 pts) Explain Your Reasoning:

3. (1 pt) The following region is rotated around a line and a student computes the volume of the resulting solid using the following integral:

$\int_c^d 2\pi (y - m)(g(y) - f(y)) \, dy$.

Which of the following best describes the axis of rotation?

a) Vertical line $x = m$, left of a
b) Vertical line $x = m$, right of b
c) Horizontal line $y = m$, below c
d) Horizontal line $y = m$, above d

(5 pts) Explain Your Reasoning:

4. (1 pt) A solid is made by rotating the region bounded inside the unit circle about the line $x = 3$. A student decides to use cylindrical shells to compute the volume. What should be used for the value of a shell radius?

a) x
b) $x - 3$
c) $3 - x$
d) $\sqrt{1 - x^2} - 3$
e) $3 - \sqrt{1 - x^2}$
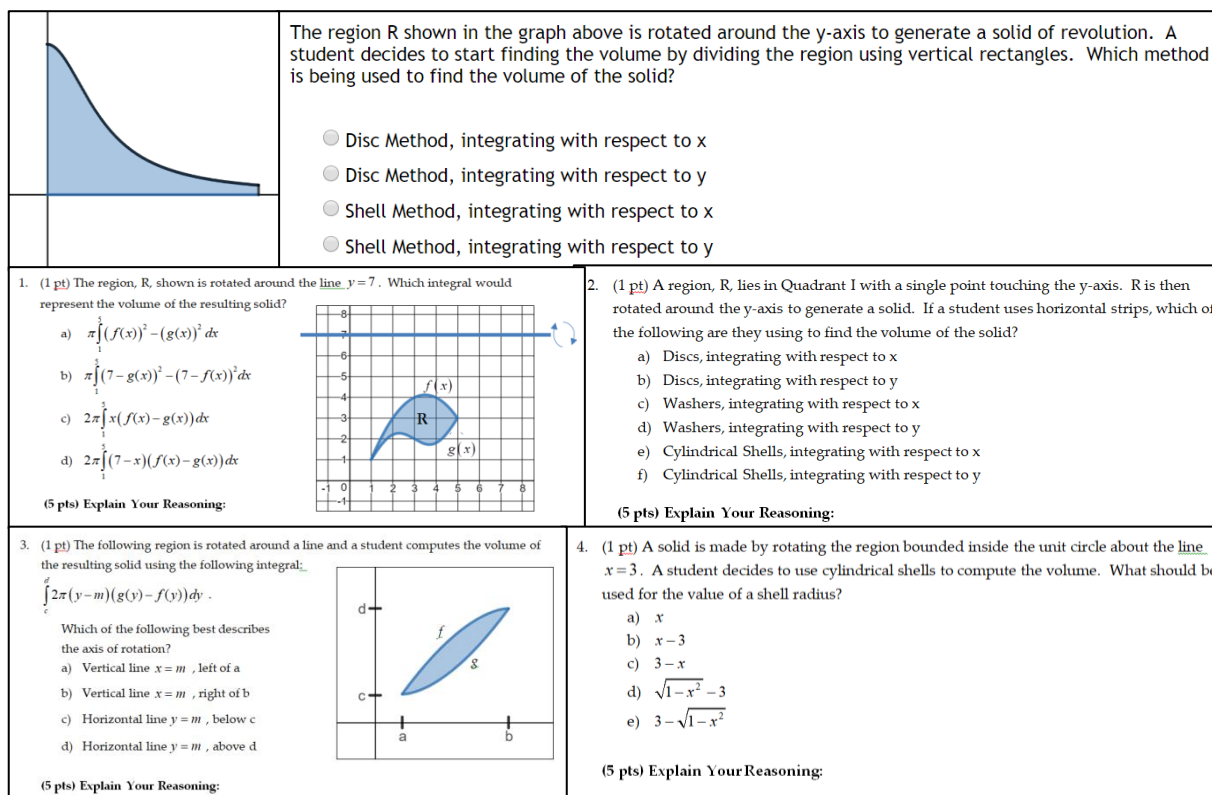
(5 pts) Explain Your Reasoning:

Figure 6: Five "Concept Check" questions. Top – CC3, the third online concept check question given prior to the third exam. Bottom four panels – Exam questions given on the third exam regarding volumes of revolution (T3-1, 2, 3, 4).

An analysis of these six questions was done to see how many students chose the correct response to the multiple choice portion of each question, and how many students provided either sound reasoning (5 on rubric), unsound reasoning (0-2 on rubric), or partially sound reasoning (3-4 on rubric). Online questions were graded only on participation, and so not awarded points using the rubric, but the responses were later evaluated in an equivalent way as sound, unsound, or partially sound. Results can be seen in Figures 7 and 8, respectively.
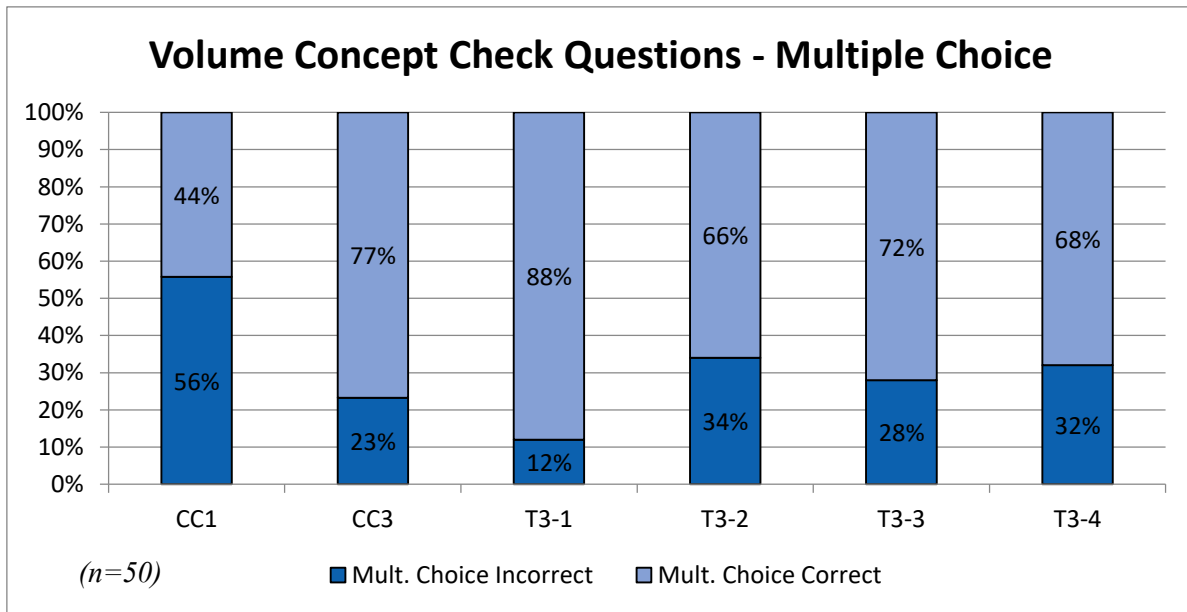
Figure 7: Accuracy of student responses to the multiple choice portion of the concept check questions for volumes or revolution.
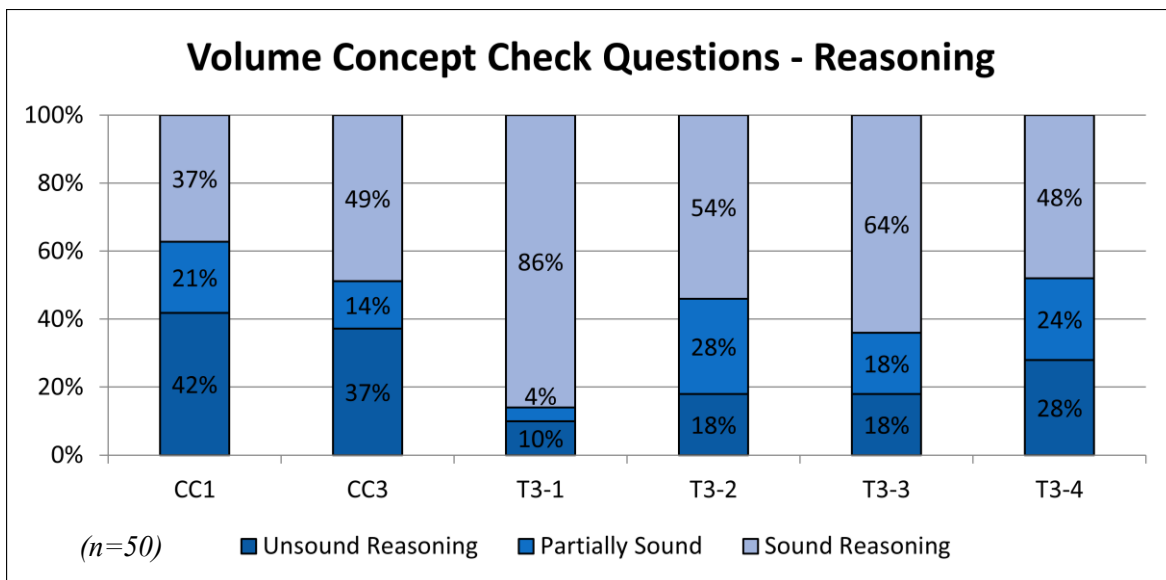
Figure 8: Reasoning results for the explanation portion of the concept check questions for volumes of revolution.

If you take a close look at the graphs, you will notice that the sound reasoning and correct multiple choice selection percentages do not match up. This is because there are usually some students that choose the correct response, but have only partially sound or even unsound reasoning. There are also students with sound or partially sound reasoning that do not choose the correct response on the multiple choice portion. Overall there was an average of 18.5% of students with unsound reasoning on the exam questions compared with an average of 39.5% on the pre-exam questions. There was also an average of 63% of students with sound reasoning on the exam versus an average of 43% on the pre-exam questions. This is likely due to the fact that the pre-exam questions were briefly addressed as a class to try and clear up any misconceptions at that point, giving students an opportunity to correct some of their own conceptual errors. Also, concept question 1 (CC1) is an indirectly related concept, with more emphasis on spatial reasoning. Since it was a new idea, it is not surprising that it showed the worst performance.

There were a few common misconceptions that may need to be addressed in the activity sheets in the future. One misconception is that if a region touches the axis of rotation at any point, you are using a disc method since the volume has no hole. This shows the student is thinking of the shape holistically rather than on a "sliced" level. Students also seemed to struggle with distances, particularly when trying to find the shell radius or washer radii when rotating around axes that were not the x or y-axis. The activities used will be updated and enhanced for future students so that these concepts are addressed with better clarity.

Student Feedback

Active learning has been shown to be effective pedagogy across STEM disciplines [20]. However, students don't always know what's good for them. For instance, students in active-learning classrooms may report that they learned less than those in passive-learning classrooms, despite exam evidence to the contrary [21]. Ideally, an intervention would provide gains in student performance as well as outstanding student reactions. Control data (from sections using similar activity sheets, but no models) is not yet available to compare performance on exam questions or final grades. However, we can still look at student responses to using the activity sheets and models in the intervention sections to get a sense of how students feel about the modified learning environment (as opposed to typical lecture classes).

Following each unit exam, students were given surveys for specific pieces of each instructional unit. In particular, they needed to respond to questions regarding the guided notes and interactive Desmos websites for the first unit, and the activity sheets with physical models for the second and third units. Each survey contained three areas of focus: communication, conceptual clarity, and connections. After the first unit survey, we increased the number of questions per area from two to three, and each question was modified to be more specific. These later surveys were administered via Google Forms. All surveys in the future will contain a similar version of these three questions per area. Students responded to each question with a 6-point Likert scale of "Completely Agree", "Somewhat Agree", "Slightly Agree", "Slightly Disagree", "Somewhat Agree", and "Completely Disagree". Response data is overwhelmingly positive for all of the unit surveys. Questions and responses for the volumes unit are shown in Figure 9.
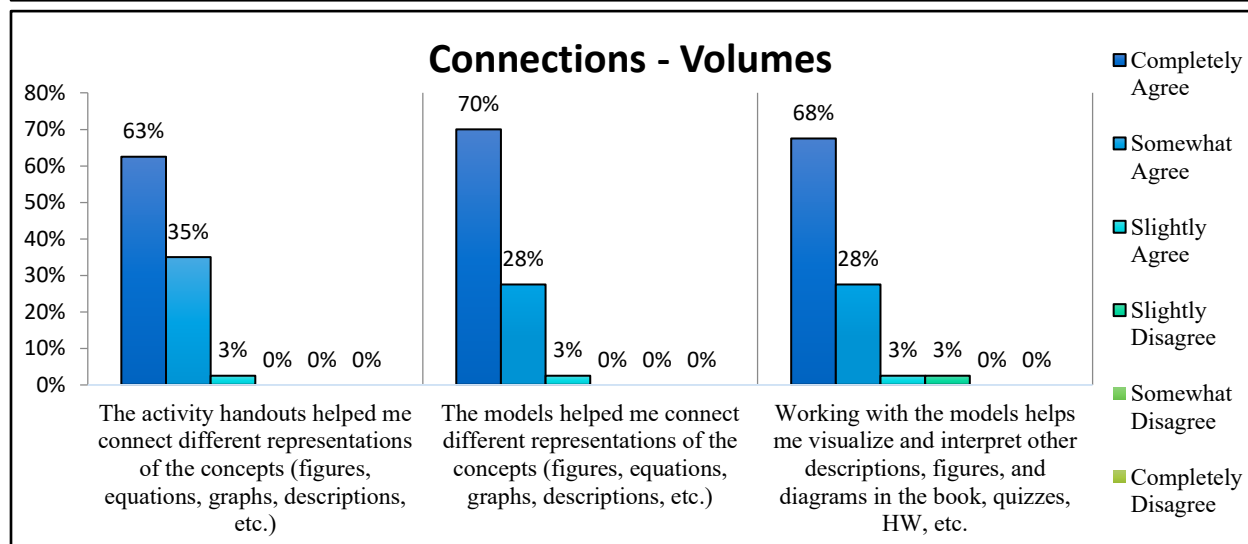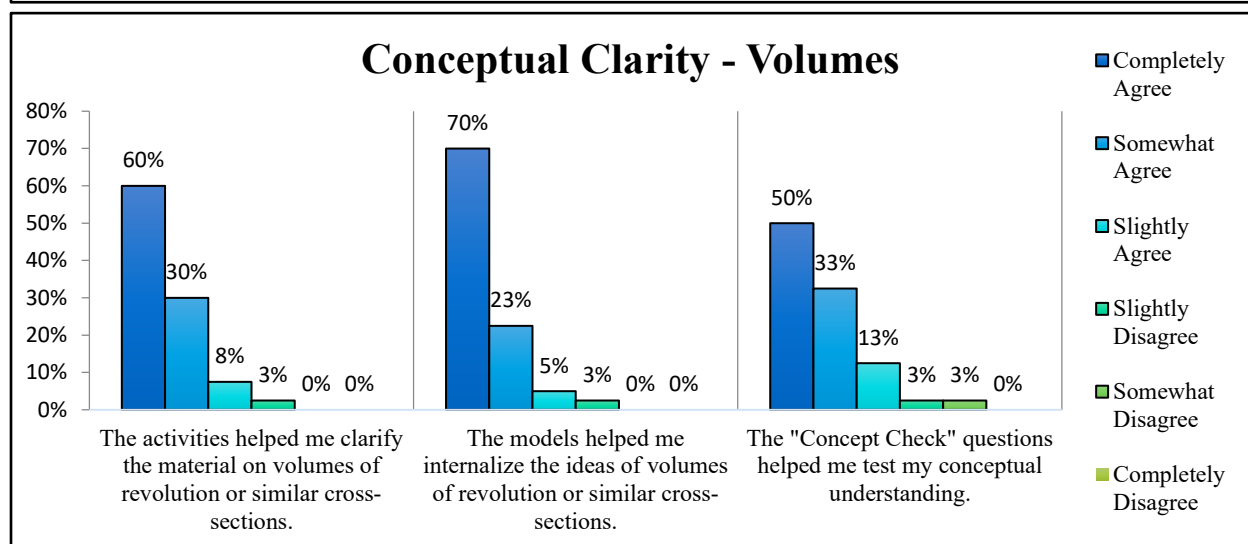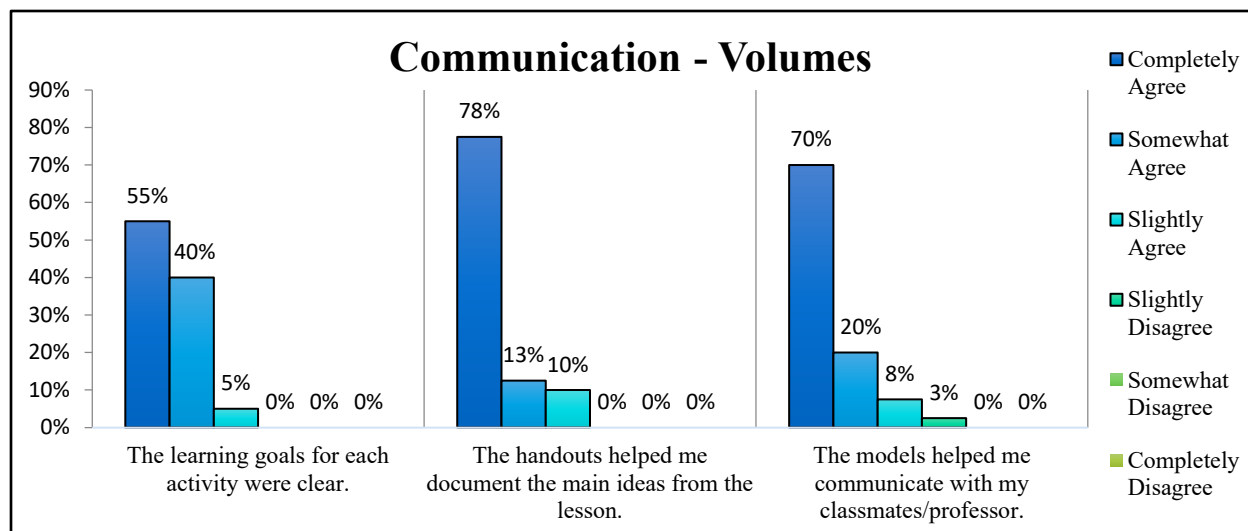
Figure 9: Student survey responses (n=40) on the use of 3D-printed physical models and activity sheets for the volumes unit.

Very similar results occurred for unit one and two as well. It seemed the concept check questions were the least popular item, although 95% of students still reported that they agreed (to various extents) that the questions helped them test their conceptual understanding. Student attitudes toward this intervention are very encouraging, especially in regard to how the handouts and models help the communication of ideas in class. More data comparisons will be done in the future after control sections have been completed.

Conclusions and Future Work

Students have a tendency to focus on step-by-step procedural rules when learning mathematics, hoping that as long as they can follow a recipe they've seen before, they can be successful. This tends to break down in upper level mathematics as well as when they arrive at other STEM courses and have to apply their mathematics to new domains and areas. Of course, they can simply try and learn new procedures, but a conceptual understanding of what is happening in math and other courses would be a much greater benefit. We have described a way of providing students with activity guides and physical models to help them develop their conceptual understanding and representational competence as they learn new material. These guides and models can help students document the main ideas of a lesson and also help them communicate ideas with their peers and professor. Providing questions to check their conceptual understanding can help students think about what they are learning not as procedural, but as a much larger conceptual body of ideas. These concept check questions can also help instructors know what students are thinking as they solve problems and be a great benefit in trying to identify common themes of misunderstanding. This type of intervention is able to be implemented at a small scale (single topic) for the wary, but can easily be expanded to more topics as an instructor becomes more comfortable with the approach.

Although active learning is considered to be superior to lecturing, small changes in the active learning environment are harder to measure. It is still unclear how much benefit the physical models have over digital representations, and future work will include a comparison of courses using physical models and those using digital or 2D-printed figures with similar activities. Course success, exam scores, performance on concept check questions, and student surveys are just some of the areas that could be analyzed for significant differences between populations. Visualization skills are also important for future engineers and can even be a predictor of success in engineering students. Comparisons of beginning and end of quarter scores on the Purdue Spatial Visualization Test – Rotations (PSVT:R) will also be analyzed for model and non-model courses.

Acknowledgements

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Appendix

Two activities are included in the pages that follow.

- The first four pages are guided notes on interpreting Riemann Sums as accumulation. This sets the stage for a more intuitive look at the Fundamental Theorem of Calculus and the Net Change Theorem. This activity uses electronic manipulatives.
- The next four pages are the student activity sheet for the introduction to solids of revolution using discs and washers. The cylindrical shell method and more practice with non-axis rotations are covered in other activities. This activity uses physical models.

Approximation Summations – Accumulation

Objectives:

- I can sketch the graph of a Reimann Sum given a number of subdivisions over a closed interval.
- I can use a Reimann Sum to approximate the change in an antiderivative function or area under a curve.

<div style="border: 2px solid; background-color: #cfe2f3; padding: 10px;">

Definition of a Riemann Sum

Let $f(x)$ be defined on a closed interval $[a,b]$ and let P be a regular partition of $[a,b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$, and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann Sum** is defined for $f(x)$ as $\sum_{i=1}^{n} f(x_i^*)\Delta x$.

</div>

We can use any point in the interval, but often we use left endpoints, right endpoints, or midpoints for the intervals.

A Riemann sum can also be interpreted as an **accumulation**. $f(x_i)$ is a slope, and Δx is the horizontal movement across the subinterval. As you move through each subinterval, following its slope for a small distance, you either raise or lower your y-value to accumulate height through the entire interval.

Net Change in y for a Reimann Sum

The open circles are the points chosen for each sub-interval. The y-values of these open points are the slopes used to go from point to point in the top curve. Each $f(x_i^*)\Delta x$ is a $\dfrac{\Delta y_i}{\Delta x} \cdot \Delta x$, which gives Δy_i for each ith sub-interval. Adding up all of the Δy_i gives the net change in y for the segmented curve.
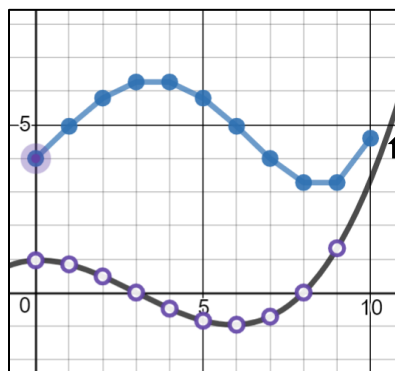
L_{10} is the Riemann sum using left endpoints on 10 subintervals.
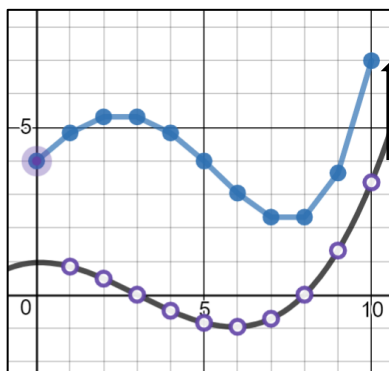
R_{10} is the Riemann sum using right endpoints on 10 subintervals.
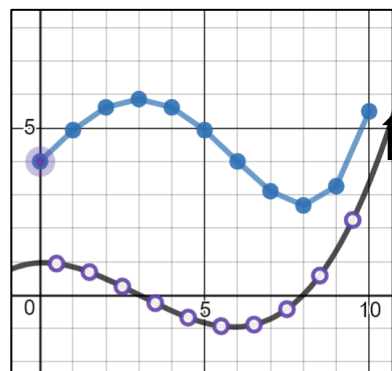
M_{10} is the Riemann sum using midpoints on 10 subintervals.

1. **Snowfall Accumulation**

   Here is a table representing the RATE of snowfall (or melt) measured in inches per day.

Time (days)	0	1	2	3	4	5	6	7	8	9
Rate of Snowfall (in / day)	4	3	1	0	-1	-2	0	1	2	-1

   a) Sketch a graph of the RATE function, then sketch a graph of the ACCUMULATION function.
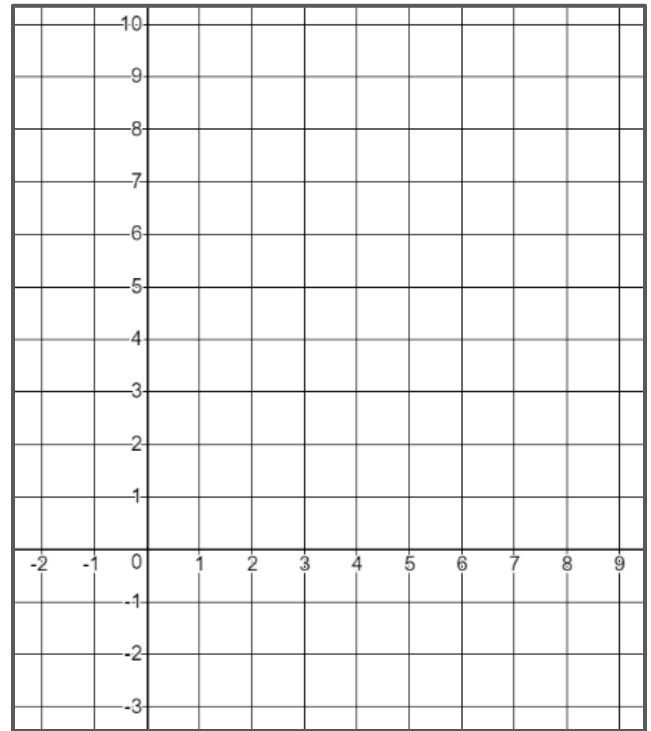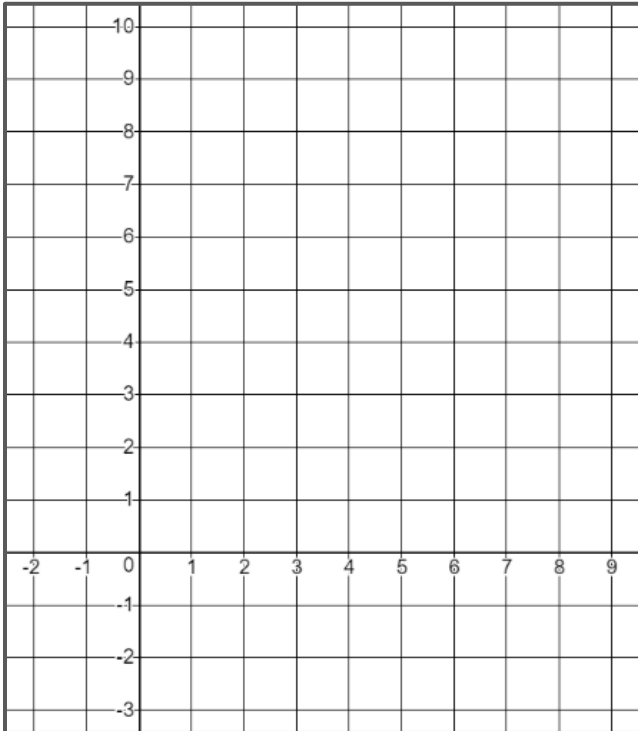
   b) When did the amount of snow (depth) appear to be the greatest?

   c) When did the amount of snow (depth) appear to be the least?

   d) What is the total accumulation of snow from the first day (t=0) through the last day (t=9)?

   e) How would you express this as a definite integral?

2. **General Accumulation Functions**

    Plot the accumulation curve formed over the interval $[-2,3]$ when you do the following:

a) Start at $(-2,-5)$ using 5 left subdivisions

b) Start at $(-2,-5)$ using 10 left subdivisions
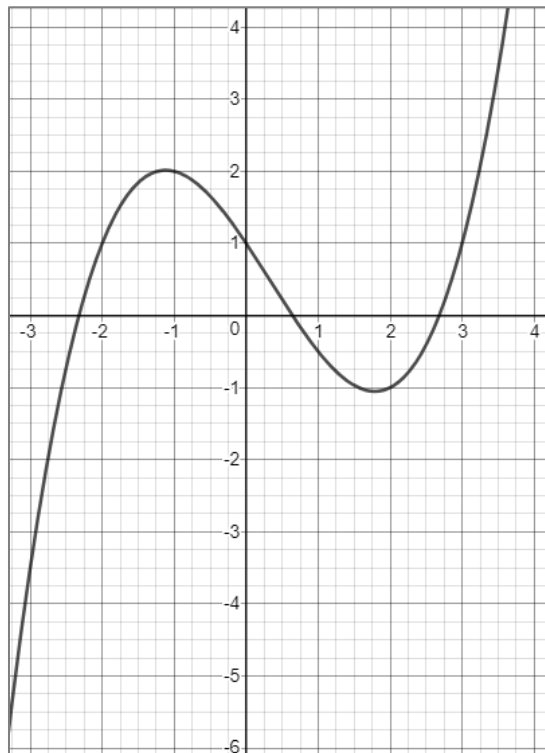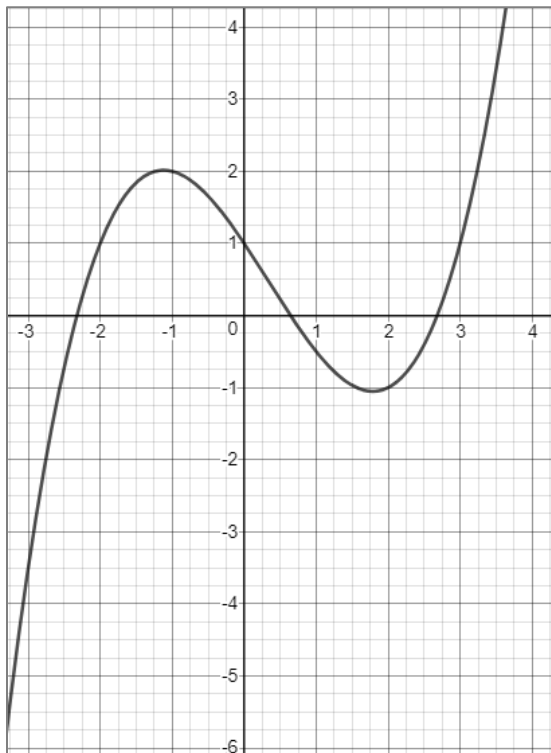
How much "y" has accumulated over the interval $[-2,3]$ in each case?

a)

b)

3. **Limits of Accumulation Functions**

    Go to the following website on your phone, tablet, or laptop.

    https://www.desmos.com/calculator/cjedeenx5f

- The black line with x's is the interval being subdivided into n segments.
- The red curve is the function f(x).
- The green dots on f(x) are the left, right, or midpoint values in each interval that tell you what slope to use in your accumulation function.
- The purple curve is the accumulation function of f(x) over $[a,b]$.
- The orange line shows you the net change in accumulated height as you move from a to b

Move the sliders around to answer the following questions.

You can:

- Drag the slider labeled "n" to change the number of subdivisions.
- Drag the slider labeled "m" to change between left, right, or midpoint calculations (m=0 is left, m=.5 is mid, m=1 is right).
- Drag the point labeled "a" left or right to change the interval bounds.
- Drag the point labeled "b" left or right to change the interval bounds.
- Drag the point labeled "start" up or down to change where your function starts.

a) What happens to the accumulation curve as the number of subdivisions on the interval gets larger and larger?

b) Compare what happens when you switch between left, mid, and right values with 5 subdivisions and with 50 subdivisions.

c) If you take the limit as your subdivisions goes to infinity, would the accumulation functions from the left, mid, and right values give you the same accumulation function or slightly different functions?

d) What might be another name for our accumulation function?

e) How could you express the net change in accumulated height using mathematical notation?

f) What happens to the accumulated height if you make a and b swap places? Which integral property does this demonstrate?

Solids of Revolution

Objectives:

- I can find the volume of a solid of revolution using discs and washers.

We will investigate several ways of computing the volume of various shaped objects. The first type of method we will explore is by rotating a region around an axis. Similar to computing area, we will start by dividing our region into several thin rectangles and looking at what happens when they are rotated around an axis. We will start with rectangles perpendicular to our axis of rotation.

Question 1:

Draw a skinny rectangle perpendicular to the line shown. Draw what it looks like rotated around the axis.

_____

Question 2:

What general formula could you use to find the volume of the rotated rectangle?

Question 3: If you added up all of the rotated rectangles in your region, what would you get for the volume of your object?

Solid #1:_____RED_____

1. Sketch the situation:
   a. What is the axis of rotation?

   b. Darken in the axis of rotation and draw a sample rectangle to rotate.

   c. SKETCH the rectangle after rotation –what shape is this?

   d. What is the volume of a general sample slice?

$$y = -(x-3)^2 + 4$$
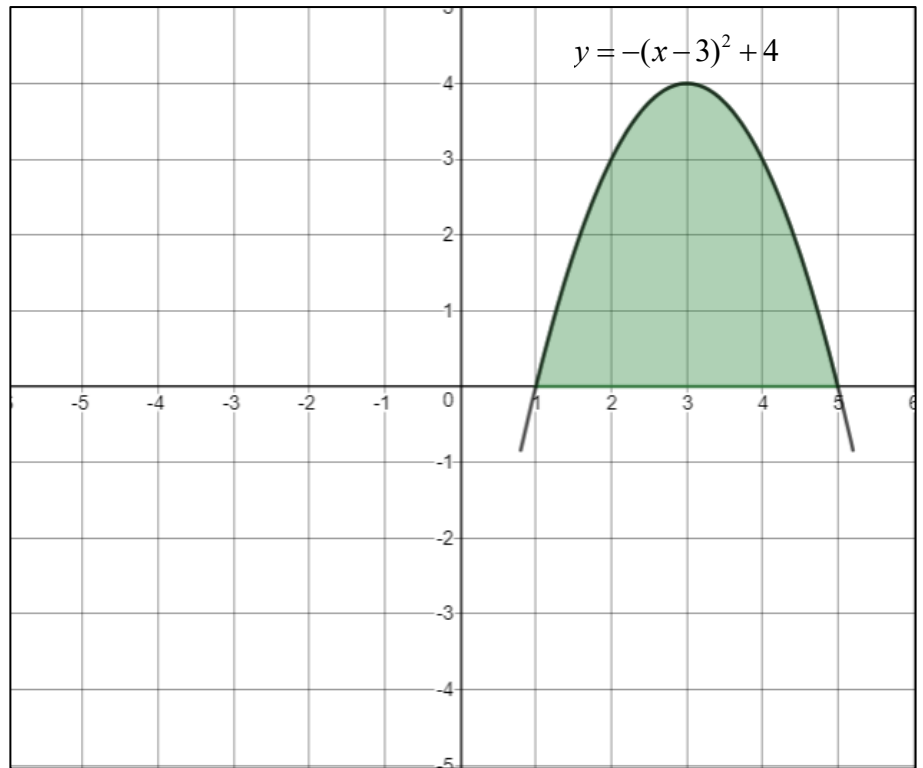
   e. Set up an integral that represents the volume using this method, then approximate the volume by evaluating the integral to 2 decimal places.
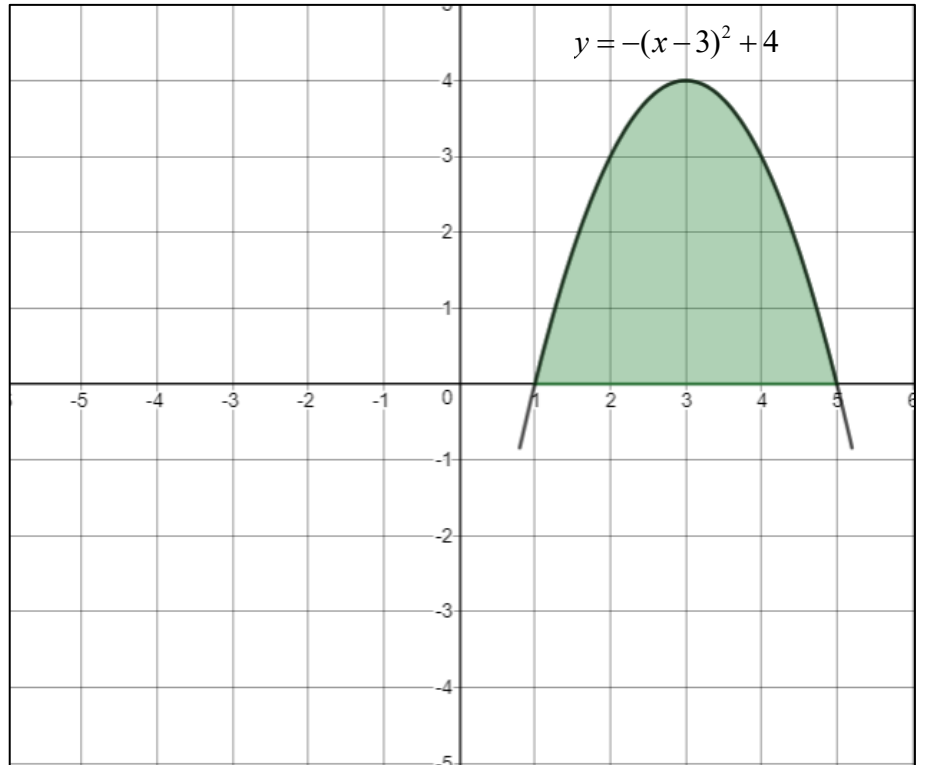
Solid #2: _____**BLUE**_____

2. Sketch the situation:
   a. What is the axis of rotation?

   b. Darken in the axis of rotation and draw a sample rectangle to rotate.

   a. SKETCH the rectangle after rotation –what shape is this?

   c. What is the volume of a general sample slice?

$$y = -(x-3)^2 + 4$$

   d. Set up an integral that represents the volume using this method, then approximate the volume by evaluating the integral to 2 decimal places.

Compare the work and end results for objects 1 and 2. What did you notice?
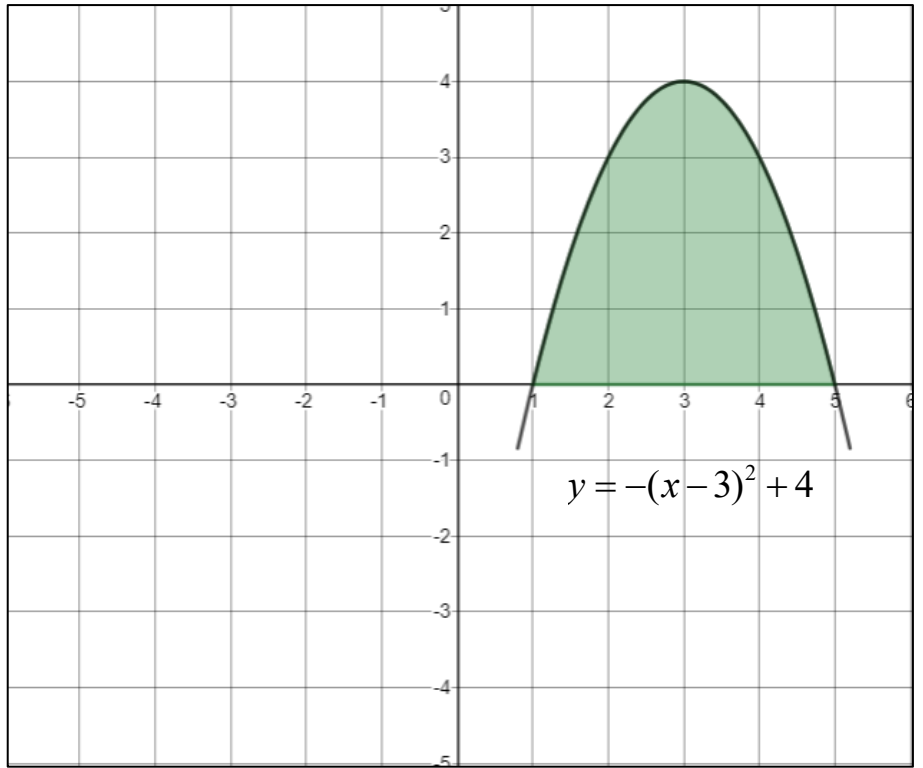
Solid #3:   TRANSPARENT

3. Sketch the situation:
    a. The axis of rotation is $y = 4$

    b. Darken in the axis of rotation and draw a sample rectangle to rotate.

    a. SKETCH the rectangle after rotation –what shape is this?

    c. What is the volume of a general sample slice?

$$y = -(x-3)^2 + 4$$

d. Set up an integral that represents the volume using this method, then approximate the volume by evaluating the integral to 2 decimal places.

Note: you can try to view the 3D object at:

- http://www.shodor.org/interactivate/activities/FunctionRevolution/ (rotatable graphs)
- https://www.wolframalpha.com/ (graph is fixed, but usually has nice graphs)